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Explicit Representation of Fundamental Units of Some Real Quadratic Fields, II

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In the preceding paper, (K. Tomita, *Proc. Japan Acad. Ser. A Sci. Math.* **71**

by using two parameters appearing in the continued fraction expansion of ω_d; In this paper, we provide analogous results for all real quadratic fields $\mathbb{Q}(\sqrt{d})$ with period 4 and 5 by using at most four parameters.

1. INTRODUCTION

For a square-free integer d congruent to 1 modulo 4, let ε_d, χ_d, and h_d be the fundamental unit, the Kronecker character, and the class number of real quadratic field $\mathbb{Q}(\sqrt{d})$, respectively. Then the Dirichlet's class number formula

$$h_d = \frac{\sqrt{d}}{2 \log \varepsilon_d} L(1, \chi_d),$$

where $L(1, \chi_d)$ is the value at $s=1$ of the L-function $L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n)\, n^{-s}$, is well known. The class number one problem for imaginary quadratic fields was solved in 1966 by A. Baker and by M. H. Stark independently (see, for example, [9]). But the problem for real quadratic fields is still open. In comparison with the imaginary case, in the real case the fundamental unit appearing in the Dirichlet formula makes it difficult to solve the problem. Namely, while in the imaginary case the unit group is a finite group, in the real case the group is infinite. Therefore, in order to study the class number problem for real quadratic fields it is very important to investigate the fundamental unit. For real quadratic fields of Richaud–Degert type the explicit form and related results of the fundamental unit

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are well known (see, for example [2, 3, 8, 15]). However, for real quadratic fields of other types they are not so well known except for [4, 12].

Generally, for a square-free positive integer d congruent to 1 modulo 4, the fundamental unit of real quadratic field $\mathbb{Q}(\sqrt{d})$ is represented by the following form:

$$\varepsilon_d = \tfrac{1}{2}(T_d + U_d\sqrt{d}) > 1, \qquad T_d,\, U_d \in \mathbb{Z}.$$

In the preceding paper [11], for all real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that the period k_d of the continued fraction expansion of $\omega_d = (1+\sqrt{d})/2$ is equal to 3, we described T_d and U_d explicitly and uniformly in the fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$ and d itself by using two parameters appearing in the continued fraction expansion of ω_d. In this paper, we provide analogous results for all real quadratic fields $\mathbb{Q}(\sqrt{d})$ with period 4 and 5. In [4], some partial quotients of continued fraction expansion of $(\sqrt{d}-1)/2$ were parameterized for the periods no greater than 4. We adopt at most four other valuable parameters in connection with [13–15] without parameterizing the partial quotients of the continued fraction expansion of ω_d.

2. MAIN RESULTS

For the set $I(d)$ of all quadratic irrational numbers in $\mathbb{Q}(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1$, $-1 < \alpha' < 0$ (α' is the conjugate of α with respect to $\mathbb{Q}$), and denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. Then, it is well known that any α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to the fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$, and that the norm of ε_d is $(-1)^{k_d}$ (see for example [10]). Moreover, in this paper the continued fraction with period k is generally denoted by $[a_0, \overline{a_1, a_2, ..., a_k}]$, and $[x]$ means the greatest integer not greater than x.

Now, for any square-free positive integer d congruent to 1 modulo 4, we can put $d = a^2 + b$, $0 < b \leqslant 2a$ $(a, b \in \mathbb{Z})$. Here, both integers a and b are uniquely determined by d because of $\sqrt{d} - 1 < a < \sqrt{d}$. Then, our main result is as follows:

THEOREM. *For a positive square-free integer d congruent to* 1 *modulo* 4, *we assume $k_d = 5$. Then, we get*

$$\omega_d = \begin{cases} [a/2, \overline{1, l, l, 1, a-1}] \text{ for an integer } l \geqslant 0 & \text{if } a \text{ is even}, \\ [(a+1)/2, \overline{l, v, v, l, a}] \text{ for two integers } l \geqslant 2,\, v > 0 & \text{if } a \text{ is odd}, \end{cases}$$

and then

$$(T_d, U_d) = \begin{cases} (A^2 r + B, A) & \text{if } a \text{ is even,} \\ (a(A^2 + l^2) + 2(vA + l), A^2 + l^2) & \text{if } a \text{ is odd} \end{cases}$$

and

$$d = A^2 r^2 + 2Br + C$$

hold, where A, B, C, and r are determined uniquely as follows:

(i) *In the case where a is even, $A = l^2 + 2l + 2$, $B = (l^2 + l)\,A + l^2$ and $C = (l^2 + 3)\,l^2 + 2(l^2 - 1)\,l + 1$. r is the non-negative integer determined uniquely by $a = Ar + l^2 + l$.*

(ii) *In the case where a is odd, $A = vl + 1$. r and s are positive integers determined uniquely by*

$$\begin{cases} Ar + ls = a, \\ lr - As = -v^2 - 1, \end{cases}$$

and $B = slA + 2v$ and $C = s(sl^2 + 4)$.

Here, we define generally the set $\mathbf{D}_t^s$ by

$$\mathbf{D}_t^s := \{ d \mid \mathbb{Z}^+ \ni d \equiv s,\, b \equiv t \pmod 8 \},$$

where $\mathbb{Z}^+$ is the set of all positive integers. Then, we obtain immediately several remarks as follows:

Remark 1. For four parameters l, v, r and s in our Theorem, we have the following result:

In the case where a is even, the pair of integers l and r in Theorem satisfy

$$(l, r) \equiv \begin{cases} (0, 0) \text{ or } (0, 2) \pmod 4 & \text{if and only if } d \in \mathbf{D}_1^1, \\ (2, 0) \text{ or } (2, 2) \pmod 4 & \text{if and only if } d \in \mathbf{D}_5^1, \\ (0, 1), (0, 3), (1, 0) \text{ or } (3, 2) \pmod 4 & \text{if and only if } d \in \mathbf{D}_1^5, \\ & \text{where } d \neq 5 \text{ i.e. } (l, r) \neq (1, 0), \\ (1, 2), (2, 1), (2, 3) \text{ or } (3, 0) \pmod 4 & \text{if and only if } d \in \mathbf{D}_5^5. \end{cases}$$

In the case where a is odd, four parameters l, v, r, and s satisfy the following conditions:

In the case of $d \equiv 1 \pmod 8$, the parameters l, v, and r satisfy $r \equiv l \not\equiv v \equiv s \equiv 0 \pmod 2$.

In the case of $d \equiv 5 \pmod 8$, the parameters l, v, r, and s satisfy $r \not\equiv l$, $rv \not\equiv s \pmod 2$.

Remark 2. For the case of $k_d = 4$, we can get the following result in the same way:

For a positive square-free integer d congruent to 1 modulo 4, we assume $k_4 = 4$.

(i) In the case where a is even, we get $\omega_d = [a/2, \overline{1, l, 1, a-1}]$ for an odd integer $l \geqslant 1$, and then $(T_d, U_d) = (A^2 r + B, A)$ and $d = A^2 r^2 + 2Br + C$ hold, where $A = l + 2$, $B = A^2 - 2$ and $C = (A+2)(A-2)$. r is the even positive integer determined uniquely by $a = Ar + A - 1$.

This case happens only for d congruent to 5 modulo 8.

(ii) In the case where a is odd, we get $\omega_d = [(a+1)/2, \overline{l, v, l, a}]$ for two integers $l, v \geqslant 1$, and then $(T_d, U_d) = (l(A+1)(Ar + sl) + 2A, l(A+1))$ and $d = (A+2)(A-2)\, r^2 + 2sl(A+2)\, r + s(sl^2 + 4)$ hold, where $A = vl + 1$. r and s are positive integers determined uniquely by $v = -r + ls$ and $a = Ar + ls$.

Remark 3. For three parameters l, r, and s, we have the following result:

In part (i) of Remark 2, the two parameters l and r satisfy

$$\begin{cases} r \equiv l - 1 \pmod 4 & \text{if and only if } d \in \mathbf{D}_1^5, \\ r \not\equiv l - 1 \pmod 4 & \text{if and only if } d \in \mathbf{D}_5^5. \end{cases}$$

In part (ii) of Remark 2, the three parameters l, r, and s satisfy

$$\begin{cases} r \equiv s \not\equiv l \equiv 0 \pmod 2 & \text{if and only if } d \in \mathbf{D}_0^1, \\ r \not\equiv s \equiv l > 1 \pmod 2 & \text{if and only if } d \in \mathbf{D}_4^5. \end{cases}$$

Remark 4. The set of all positive square-free integers congruent to 1 modulo 8 is the union of $\mathbf{D}_1^1$, $\mathbf{D}_5^1$, $\mathbf{D}_0^1$, and the set of all positive square-free integers congruent to 5 modulo 8 is the union of $\mathbf{D}_1^5$, $\mathbf{D}_5^5$, $\mathbf{D}_4^5$.

The above six sets are represented as follows:

$$\mathbf{D}_1^1 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m + 1,\ a \equiv 0 \pmod 4,\ 0 \leqslant 4m < a\}$$

$$\mathbf{D}_5^1 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m + 5,\ a \equiv 2 \pmod 4,\ 0 \leqslant 4m < a - 2\}$$

$$\mathbf{D}_0^1 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m,\ a \equiv 1 \pmod 2,\ 0 < 4m < a\}$$

$$\mathbf{D}_1^5 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m + 1,\ a \equiv 2 \pmod 4,\ 0 \leqslant 4m < a\}$$

$$\mathbf{D}_5^5 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m + 5,\ a \equiv 0 \pmod 4,\ 0 \leqslant 4m < a - 2\}$$

$$\mathbf{D}_4^5 := \{d \mid \mathbb{Z}^+ \ni d = a^2 + 8m + 4,\ a \equiv 1 \pmod 2,\ 0 \leqslant 4m < a - 2\}.$$

3. PROOF OF THEOREM

For the set $\mathbf{D}$ of all square-free positive integers, we define the set $I_k(\mathbf{D})$ by

$$I_k(\mathbf{D}) := \{\omega_d \mid d \in \mathbf{D} \text{ and the period of } \omega_d = (1+\sqrt{d})/2 \text{ is } k\},$$

and we put $\omega_0 := q_0 - 1 + \omega$ for $\omega = [q_0, \overline{q_1, ..., q_k}]$ in $I_k(\mathbf{D})$. Then, ω_0 belongs to $R(d)$. For ω_0 in $R(d)$, let $\omega_i = l_i + 1/\omega_{i+1}$ ($l_1 = [\omega_i]$, $i \geqslant 0$) be the continued fraction expansion of ω_0. Moreover, each ω_0 is expressed in the form $\omega_i = (a - r_i + \sqrt{d})/c_i$ ($c_i, r_i \in \mathbb{Z}$) from Lemma 2 in [11].

Proof of Theorem. In the case where a is even, we first assume that ω belongs to $I_5(\mathbf{D}_1^1 \cup \mathbf{D}_1^5)$. Then $[\omega] = q_0 = a/2$ holds, and Lemma 3 in [11] implies $r_0 = r_1 = a - l_0 = 1$. Hence, it follows from $c_0 = 2$ that $c_1 = (8m + 1 + 2a - 1)/2 = 4m + a$. Since Lemma 2 in [11] implies $2a - r_1 = c_1 l_1 + r_2$, it holds that $2a - 1 = (4m + a)\, l_1 + r_2$. Hence, we get $(2 - l_1)\, a = 4ml_1 + r_2 + 1 > 0$. Here, $l_1 \geqslant 1$ holds, and $a > 0$ implies $l_1 \leqslant 2$. Hence, if we assume $l_1 = 2$, then we get $8m = -r_2 - 1$. However, $r_2 \geqslant 0$ implies $m \leqslant 0$, which is a contradiction. Therefore we get $l_1 = 1$, and so because of $l_2 = l_{5-2} = l_3$ and $l_4 = l_1 = 1$, we have $\omega = [a/2, \overline{1, l_2, l_2, 1, a-1}]$. Since $2a - 1 = (4m + a)\, l_1 + r_2$ and $l_1 = 1$, we get

$$a = 4m + r_2 + 1. \tag{1}$$

It follows by Lemma 2 in [11] that $2a - r_3 = c_2 l_2 + r_2$ and $c_3 = c_1 + (r_3 - r_2)\, l_2 = c_2 = r_2 + 1$. Hence we get

$$2a = (r_2 + 1)\, l_2 + r_2 + r_3 \qquad \text{and} \qquad 4m + a + (r_3 - r_2)\, l_2 = r_2 + 1, \tag{2}$$

because of $c_1 = 4m + a$. Therefore, from (2), we get

$$8m = (r_2 - 2r_3 - 1)\, l_2 + r_2 - r_3 + 2. \tag{3}$$

On the other hand, from (1) and (2), we get

$$8m = (r_2 - r_3)\, l_2, \tag{4}$$

and hence from (3) and (4) we get

$$r_2 = (r_3 + 1)\, l_2 + r_3 - 2. \tag{5}$$

Moreover, (4) and (5) imply

$$8m = (r_3 + 1)\, l_2^2 - 2l_2. \tag{6}$$

If we assume that ω belongs to $I_5(\mathbf{D}_1^1)$, then since $a \equiv 0 \pmod 4$ in (1), we get $r_2 \equiv 3 \pmod 4$, from which $r_3 \equiv 1 \pmod 2$ holds by (2). Similarly, if we assume that ω belongs to $I_5(\mathbf{D}_1^5)$, then since $a \equiv 2 \pmod 4$ in (1), we get $r_2 \equiv 1 \pmod 4$, and so get also $r_3 \equiv 1 \pmod 2$ by (2). Hence, we can put $l_2 = l$, $r_3 = 2r+1$ for two integers l, r, and get $a = (l^2+2l+2)\,r + l^2 + l$ from (2), (5). Here, r is non-negative because of $r_i \geqslant 0$ in Lemma 2 in [11]. Moreover, if we put $A = l^2+2l+2$, $B = (l^2+l)\,A + l^2$ and $C = (l^2+3)\,l^2 + 2(l^2-1)\,l+1$, then we get $d = A^2r^2 + 2Br + C$, and also in Lemma 1 in [11] we get $Q_4 = l^2+l+1$ and $Q_5 = A$ because of $\omega = [a/2, \overline{1, l, l, 1, a-1}]$. Hence, we have $T_d = A^2r + B$, $U_d = A$. In the case of ω in $I_5(\mathbf{D}_5^1 \cup \mathbf{D}_5^5)$, it obtains in a way similar to that for the case of $I_5(\mathbf{D}_1^1 \cup \mathbf{D}_1^5)$.

In the case where a is odd, we have only to consider d in $\mathbf{D}_0^1 \cup \mathbf{D}_4^5$. It follows from $q_0 = [\omega] = (a+1)/2$ and Lemma 3 in [11] that $r_0 = r_1 = a - l_0 = 0$ and $c_0 = 2$. We assume that ω belongs to $I_5(\mathbf{D}_0^1)$. Then $c_1 = 4m$ holds and Lemma 2 in [11] implies $2a = 4ml_1 + r_2$. Hence, we can put $r_2 = 2r$ and get $a = 2ml_1 + r$. Moreover, from Lemma 2 in [11] we get $c_2 = r_2 l_1 + 2$ and $2a = c_2 l_2 + r_2 + r_3$, and hence we get

$$4ml_1 = 2(rl_1+1)\,l_2 + r_3. \tag{7}$$

On the other hand, $c_2 = r_2 l_1 + 2$ and $c_3 = c_1 + (r_3 - r_2)\,l_2$ imply

$$4m = (2r - r_3)\,l_2 + 2(rl_1+1). \tag{8}$$

because of $c_3 = c_{5-3} = c_2$. If we assume $l_2 \equiv 1 \pmod 2$, then in the case of $l_1 \equiv 1 \pmod 2$ we get $r_3 \equiv 0 \pmod 4$ from (7). Hence $4m = 2 \pmod 4$ holds in (7), which is a contradiction. In the case that $l_1 \equiv 0 \pmod 2$, we also get $4m \equiv 2 \pmod 4$. Hence, we have $l_2 \equiv 0 \pmod 2$. Therefore, from (7), (8), we can determine $r_3 = 0 \pmod 4$ and $l_1 \equiv 1 \pmod 2$, respectively. Moreover, from (7), $2l_2 + r_3 = 0 \pmod{l_1}$ holds. Accordingly, we can put $l = l_1$ for an odd integer l and $v = l_2$ for an even integer v, respectively, and there exists a positive even integer s such that $r_3 = 2(sl - v)$, because of $r_3 \equiv 0 \pmod 4$. By substitution of this r_3 in (7), we get $2m = rv + s$, and because of $a = 2ml + r$ we get $(vl+1)\,r + ls = a$. On the other hand, (8) implies $2(rv+s) = \{2r - 2(sl-v)\} + 2(rl+1)$, and hence we get $lr - (vl+1)\,s = -v^2 - 1$. Therefore, because of $(vl+1)^2 - l^2 \neq 0$, such integers r, s are uniquely determined.

Now, we put $A = vl+1$. Then, since $\omega = [(a+1)/2, \overline{l, v, v, l, a}]$, $Q_4 = vA + l$ and $Q_5 = A^2 + l^2$ hold in Lemma 1 in [11]. Therefore, we have that

$$\begin{cases} T_d = (Ar+sl)(A^2+l^2) + 2(vA+l), \\ U_d = A^2 + l^2. \end{cases}$$

Moreover, if we put $B = slA + 2v$ and $C = s(sl^2+4)$, then $d = A^2r^2 + 2Br + C$ holds.

Next, we assume that ω belongs to $I_5(\mathbf{D}_4^5)$. Then, we have only to replace $8m$ with $8m+4$ in the case that ω belongs to $I_5(\mathbf{D}_0^1)$. Hence, (7) and (8) are replaced by $(4m+2)\, l_1 = 2(rl_1+1)\, l_2 + r_3$ and $4m+2 = (2r - r_3)\, l_2 + 2(rl_1+1)$, respectively. Then, there exists the positive integer s satisfying $r_3 = 2(sl_1 - l_2)$. The remaining part of this case is proved in the same way as the previous case. Thus, the Theorem is completely proved. ∎

4. APPLICATIONS

For any square-free positive integer d, in [14, 15] Yokoi defined some new invariants by taking the fundamental unit of $\mathbb{Q}(\sqrt{d})$ as

$$n_d := \left[\frac{T_d}{U_d^2}\right], \qquad m_d := \left[\frac{U_d^2}{T_d}\right], \text{ etc.,}$$

and studied the relationship between these new invariants and already defined invariants such as the class number. (For the invariant n_d there is another definition in [8].)

In this section, we apply our results to these invariants, and consider the class number of the real quadratic fields $\mathbb{Q}(\sqrt{d})$ for d in $\mathbf{D}^1$ and $\mathbf{D}^5$, where $\mathbf{D}^s$ is the set of all positive square-free integers congruent to s modulo 8.

Corollary 1. *Assume $k_d = 4$. Then, under the notation of Remark 2. $m_d \neq 0$ holds if and only if $d = (A+2)(A-2)\, r^2 + 2sl(A+2)\, r + s(sl^2+4)$ and $r < 1$.*

Moreover, if a is even, then n_d is also even positive integer and $m_d = 0$ holds.

Proof. In the case of even a, by Remarks 2 and 3 we get first $d \in \mathbf{D}_1^5 \cup \mathbf{D}_5^5$. Since $B < A^2$, we next get $n_d = [(A^2r + B)/A^2] = r$, which implies that n_d is even and $n_d \neq 0$ because of $r \neq 0$.

In the case of odd a, we get $n_d = [(Ar + sl)/l(A+1)]$. Here, we consider a condition for $Ar + sl < l(A+1)$. In the case of $r < l$, $A - l(s-1) = l\{s(l-1) - r + 1\} + 1$ and $r \leqslant l - 1 < s(l-1) + 1$ imply $Ar + sl - l(A+1) < 0$. On the other hand, in the case of $r > l$, it follows from $sl - r > 0$ that s is positive integer. Since $A > 0$ and $Ar + sl - l(A+1) > 0$, we have $n_d > 1$. Therefore, if $n_d = 0$, then $r < l$ holds. This is the necessary and sufficient condition for $m_d \neq 0$. ∎

COROLLARY 2. *Assume that $k_d = 5$ and a is even. Then, $m_d = 1$ holds if and only if*

$$d = (l^2 + 3)\, l^2 + 2(l^2 - 1)\, l + 1 \qquad (l \in \mathbb{Z}^+, l \neq 1).$$

In this case, d belongs to the set $\mathbf{D}_1^1$, $\mathbf{D}_1^5$, $\mathbf{D}_5^1$, $\mathbf{D}_5^5$ corresponding to $l \equiv 0, 1, 2, 3 \pmod 4$ respectively.

Proof. In Theorem, since $A^2 - B = (l+2)\, A + l^2 > 0$, $n_d = [(A^2 r + B)/A^2] = r \geqslant 0$ holds. Hence, $m_d \neq 0$ if and only if $r = 0$ holds. Moreover, $A^2 - B < B$ implies $0 < m_d = [A^2/(A^2 r + B)] = [A^2/B] < 2$. Therefore, $m_d = 1$ holds, and if $r = 0$, then $d = C$. Conversely, if we put $d = C$, then $r = 0$ holds, from which Corollary 2 follows immediately. ∎

COROLLARY 3. *Assume that $k_d = 5$ and a is odd. Then, under the notation of Theorem, $m_d \neq 0$ holds if and only if $a < U_d = A^2 + l^2$, and when these conditions are satisfied, $d = A^2 r^2 + 2Br + C$ holds.*

Proof. Since $Q_i < Q_{i+1}$ in Lemma 1 of [11], if $m_d \neq 0$ then $2q_0 - 1 < U_d = Q_5$. In this case we get $q_0 = (a+1)/2$, from which $a < U_d$ holds.

On the other hand, in Theorem, we assume $U_d > a$. Since $U_d - 2(vA + l) = A(A - 2v) + l(l-2)$ and $l \geqslant 2$, we get $U_d > 2(vA + l)$. Hence, we get also $n_d = [(a(A^2 + l^2) + 2(vA + l))/(A^2 + l^2)^2] = [a/(A^2 + l^2)] < 0$. Therefore, $a < U_d$ is necessary and sufficient condition for $m_d \neq 0$.

COROLLARY 4. *Assume that $k_d = 4$ and a is even. Then, there exist exactly the following five d's such that real quadratic fields $\mathbb{Q}(\sqrt{d})$ have the class number one*:

$$d = 213, 717\ (\in \mathbf{D}_1^5), 69, 413, 1077\ (\in \mathbf{D}_5^5).$$

Proof. By Corollary 1, we know $n_d \neq 0$. Therefore, Corollary 4 follows immediately from the Table III in [13]. ∎

Remark 5. Assume that $k_d = 4$ and a is odd. Then it is known that for d in $\mathbf{D}_0^1$ the class number of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is equal to one if and only if $d = 33$ (see [5]), and moreover that if d is in $\mathbf{D}_4^5$ and is less than or equal to 50,000, then there exist exactly the following seven d's such that real quadratic fields $\mathbb{Q}(\sqrt{d})$ have the class number one (see [7]):

$$d = 133, 141, 573, 1293, 1397, 1757, 3053.$$

COROLLARY 5. *Assume that $k_d = 5$ and a is even. Then, there exist exactly the following five d's such that real quadratic fields $\mathbb{Q}(\sqrt{d})$ have the class number one with one possible exception of d*:

$$d = 41\ (\in \mathbf{D}_5^1), 941\ (\in \mathbf{D}_1^5), 149, 157, 269\ (\in \mathbf{D}_5^5).$$

Proof. For d in $\mathbf{D}_0^1$, we know that there exists only one real quadratic fields $\mathbb{Q}(\sqrt{d})$ with $d=41=6^2+5\in\mathbf{D}_5^1$ (see [6]). On the other hand, by [13], we know that there exist exactly two discriminants $d=149, 269\in\mathbf{D}_5^5$ of real quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying both $n_d\neq 0$ and $h_d=1$.

In the case of $m_d\neq 0$, namely $n_d=0$, it follows from [14, 15] that

$$h_d>0.3275s^{-1}d^{(s-1)/2s}/\log(m_d+1)\,d$$

for $s=11.2$, $d\geqslant 73131$ and with one possible exception of d. Hence, since $m_d=1$ in Corollary 2, we get

$$h_d>\frac{0.3275d^{0.41071}}{11.2\log 2d},$$

and since $h_d>1$ holds for any $d\geqslant 4679055$, it is enough to consider only the positive square-free integer d satisfying $d<4679055$. From Corollary 2, such d are $\{$41, 157, 941, 3221, 5297, 8245, 12,281, 17,645, 24,601, 33,437, 44,465, 94,181, 117,577, 145,085, 177,161, 214,285, 256,961, 305,717, 361,105, 423,701, 494,105, 572,941, 660,857, 866,641, 1,117,121, 1,260,997, 1,418,345, 1,589,981, 1,776,745, 1,979,501, 2,199,137, 2,436,565, 2,692,721, 2,968,565, 3,265,081, 3,583,277, 3,924,185, 4,288,861, 4,678,385$\}$. Therefore, from calculation by computer,[1] we know that among them there exist exactly three discriminants $d=41$, 157, 941 of real quadratic fields $\mathbb{Q}(\sqrt{d})$ with $h_d=1$. ∎

Remark 6. Assume that $k_d=5$ and a is odd. It is known that if d is less than or equal to 50,000, then there exist exactly the following seven d's such that real quadratic fields $\mathbb{Q}(\sqrt{d})$ have the class number one (see [7]):

$$d=181, 397, 1013, 2477, 2693, 3533, 4253\ (\in\mathbf{D}_4^5).$$

For a prime p congruent to 1 modulo 4, the following conjecture on the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{p})$ is well known (see [1]):

Conjecture (Ankney–Artin–Chowla). Let p be a prime congruent to 1 modulo 4. Then, for the fundamental unit $\varepsilon_p=(T_p+U_p)/2>1$, $(T_p, U_p\in\mathbb{Z}^+)$ of the real quadratic field $\mathbb{Q}(\sqrt{p})$, $U_p\not\equiv 0 \pmod p$ holds.

Moreover, it is conjectured that $U_d\not\equiv 0 \pmod d$ holds for positive square-free integer d congruent to 7 modulo 8 (see [8]).

Corollary 6. *Let p be an odd prime congruent to* 1 *modulo* 4. *If $k_p=4$, then the conjecture of Ankney, Artin, and Chowla is true.*

If $k_p=5$ and a is odd, then the conjecture is also true.

[1] We use C libraries of PARI-GP.

Proof. In the case where a is even and $k_p = 4$, by Corollary 1 $n_p \neq 0$ holds, from which it follows that the Ankney–Artin–Chowla conjecture is true (see [14]).

In the case where a is odd and $k_p = 4$, by Remark 2

$$p - U_p = (A+2)(A-2)\,r^2 + l\{(2sr-1)(A+1) + 2sr\} + s(sl^2 + 4)$$

and $2sr > 1$ hold, because $p > 0$, $U_p > 0$ and s, r are positive integers. Hence, we get $p > U_p$, and so the conjecture is true.

Finally, in the case where a is odd and $k_p = 5$, we also get $p > A^2 + C > U_p \neq 0$ by the Theorem. Thus, Corollary 6 is proved completely. ∎

For d in $\mathbf{D}_0^1$ and $\mathbf{D}_4^5$ these results are more complicated than the others. Therefore, we consider only the positive square-free integers d with class number one, and provide Table I, which shows the distribution of the

TABLE I

The Distribution of the Number of d's Such That $d < 50{,}000$, $k_d \leqslant 24$, and $h_d = 1$

k_d	$\#(\mathbf{D}_1^1)$	$\#(\mathbf{D}_5^1)$	$\#(\mathbf{D}_0^1)$	$\#(\mathbf{D}_1^5)$	$\#(\mathbf{D}_5^5)$	$\#(\mathbf{D}_4^5)$
1				1		5
2				1	2	5
3	1			4		6
4			1	2	3	7
5		1		1	3	7
6			1	1	1	9
7		1	2	4	3	7
8				3	4	6
9	1	1	2	2	8	7
10	1		1	3	3	8
11	1	1	1	3	8	10
12		1	1	6	1	13
13			3	7	6	15
14		3		5	7	9
15	1		1	5	3	8
16		2	3	3	9	10
17	1	1	1	9	7	8
18	2	2	3	9	4	26
19		2	4	4	8	16
20			1	5	6	14
21		1	2	5	5	17
22		1	3	10	9	18
23	2		4	4	6	21
24		3	3	8	10	24
Total	10	20	37	105	116	276

number of such d's ($<50,000$) with each k_d ($\leqslant 24$), where $\#(\mathbf{D}_i^j)$ in the table denotes the number of such d's belonging to the set $\mathbf{D}_i^j$. For example, in $k_d=4$, there exist exactly 1, 2, 3, and 7 such d's belonging to the sets $\mathbf{D}_0^1$, $\mathbf{D}_1^5$, $\mathbf{D}_5^5$, $\mathbf{D}_4^5$, respectively (see Corollary 4, Remark 5 and [7]). In $\mathbf{D}^5$ there exist exactly 497 such d's, and among them 276 d's belong to the set $\mathbf{D}_4^5$. In $\mathbf{D}^1$ there exist exactly 67 such d's, and among them 37 d's belong to the set $\mathbf{D}_0^1$.

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